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STABILITY OF STEADY MOTIONS OF A RIGID BODY WITH AN ELASTIC SHELL PARTIALLY FILLED WITH FLUID* (**)

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A problem of stability of steady motions of a rigid body with a cavity in the form of a closed thin elastic shell partially filled with fluid, in a conservative force field, is considered. It is assumed that stationary holonomic constraints are imposed on the body allowing its rotation about some spatially fixed axis, and the forces acting on the body have zero moment about this axis. The conditions of stability are obtained from the solution of the problem dealing with the minimum of the changed potential energy W of the system obtained by studying its second variation. Sufficient conditions of the positive definiteness of $\delta^2 W$ are obtained in the form of Silvester condition of positive definiteness of some quadratic form of a finite number of variables. A method for constructing this quadratic form is given.

1. Let us consider a motion of a rigid body with a cavity in the form of a closed, thin elastic shell partially filled with fluid, the surface tension of which can be neglected, in a potential force field. We assume that stationary constraints imposed on the body allow its rotation about the ξ_3-axis of the inertial rectangular coordinate system $O'\xi_1\xi_2\xi_3$, and the forces acting on the body have zero moment about this axis. We also introduce a moving rectangular coordinate system $Ox_1x_2x_3$ the unit vectors $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ of which coincide with the principal central axes of inertia of the body and shell in the undeformed state. The position of the rigid body relative to the $O'\xi_1\xi_2\xi_3$ coordinate system will be described by the Lagrangian coordinates $q_1, \ldots, q_n$ $(n \leqslant 6)$ where q_n is the angle of rotation of the body about the ξ_3-axis. We define the middle surface S of the shell in the undeformed state by the equation /1/

$$\mathbf{M}(\alpha+2\pi, \beta) = \mathbf{M}(\alpha, \beta) = \sum_{\nu=1}^{3} x_\nu(\alpha, \beta)\,\mathbf{i}_\nu \qquad (1.1)$$
$$(0 \leqslant \alpha < 2\pi, \quad \beta_1 \leqslant \beta \leqslant \beta_2)$$

where α and β are coordinates of a point in the surface. We take the lines of curvature of the middle surface as coordinate lines $\alpha = \text{const}$ (β-lines) and $\beta = \text{const}$ (α-lines) and assume that the α-lines are closed and the values $\beta = \beta_1$, $\beta = \beta_2$ correspond to the edges of the shell. We introduce a triad of vectors $\mathbf{M}_\alpha, \mathbf{M}_\beta, \mathbf{n}$

$$\mathbf{M}_\alpha = \frac{\partial \mathbf{M}}{\partial \alpha}, \quad \mathbf{M}_\beta = \frac{\partial \mathbf{M}}{\partial \beta}, \quad \mathbf{n} = \frac{1}{AB}(\mathbf{M}_\alpha \times \mathbf{M}_\beta) \qquad (1.2)$$
$$\frac{\mathbf{M}_\alpha}{A} \times \mathbf{n} = -\frac{\mathbf{M}_\beta}{B}, \quad \frac{\mathbf{M}_\beta}{B} \times \mathbf{n} = \frac{\mathbf{M}_\alpha}{A}, \quad A^2 = \mathbf{M}_\alpha^2, \quad B^2 = \mathbf{M}_\beta^2$$

We adopt, for the three-dimensional space occupied by the shell, the Kirchhoff—Love /1/ hypothesis on the conservation of the normal element. Then the regions occupied by the shell in the undeformed and deformed state can be described, respectively, by the equations /1/

$$\mathbf{M}^* = \mathbf{M}(\alpha, \beta) + z\mathbf{n}(\alpha, \beta), \quad \mathbf{M}^{*\prime} = \mathbf{M}^*(\alpha, \beta) + \mathbf{U} + z(\mathbf{n} \times \mathbf{\Omega}) \qquad (1.3)$$
$$-h \leqslant z \leqslant h$$
$$\mathbf{U}(t, \alpha+2\pi, \beta) = \mathbf{U}(t, \alpha, \beta) = u(t, \alpha, \beta)\frac{\mathbf{M}_\alpha}{A} + v(t, \alpha, \beta)\frac{\mathbf{M}_\beta}{B} - w(t, \alpha, \beta)\,\mathbf{n}$$
$$\mathbf{\Omega}(t, \alpha, \beta) = \gamma\frac{\mathbf{M}_\beta}{B} - \gamma'\frac{\mathbf{M}_\alpha}{A} + \gamma''\mathbf{n}$$
$$\gamma = -\left(\frac{1}{A}\frac{\partial w}{\partial \alpha} + \frac{u}{R_1}\right), \quad \gamma' = -\left(\frac{1}{B}\frac{\partial w}{\partial \beta} + \frac{v}{R_2}\right)$$
$$\gamma'' = \frac{1}{AB}\left[\frac{\partial}{\partial \beta}(Au) - \frac{\partial}{\partial \alpha}(Bv)\right]$$

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where $2h$ denotes the shell thickness, U is the middle surface elastic displacement vector and Ω is the vector of elastic rotation. We denote by x_ν^*, $x_\nu^{*\prime}$ ($\nu = 1, 2, 3$) the coordinates of the points of the space occupied by the shell in the undeformed and deformed state, respectivly. Then from (1.1)—(1.3) we obtain

$$x_1^* = x_1(\alpha, \beta) + \frac{z}{AB}(x_{2\alpha}x_{3\beta} - x_{3\alpha}x_{2\beta}), \quad x_1^{*\prime} = x_1^* + w_1 \tag{1.4}$$

$$w_1 = w_1^{(0)} - zw_1^{(1)}, \quad w_1^{(0)} = \frac{x_{1\alpha}}{A}u + \frac{x_{1\beta}}{B}v - n_1 w \ (1\,2\,3)$$

$$w_1^{(1)} = \frac{x_{1\alpha}}{A}\gamma + \frac{x_{1\beta}}{B}\gamma', \quad n_1 = \frac{1}{AB}(x_{2\alpha}x_{3\beta} - x_{3\alpha}x_{2\beta})$$

where n_1, n_2 and n_3 are projections of the vector $\mathbf{n}$ on the x_1, x_2 and x_3 axes.

Let us assume that the shell is rigidly clamped along its edges to the rigid lids (or bulkheads) situated at a constant distance from each other, so that

$$u = v = w = 0, \quad \partial w/\partial\beta = 0 \text{ as } \beta = \beta_1, \beta = \beta_2, 0 \leqslant \alpha < 2\pi \tag{1.5}$$

We adopt the following expression /1/ for the potential energy of shell deformation:

$$\Pi_d = \frac{2Eh}{1-\sigma^2}\int_S \Pi_*(\varepsilon, \varkappa)\, AB\, d\alpha\, d\beta \tag{1.6}$$

$$2\Pi_* = (\varepsilon_1 + \varepsilon_2)^2 - 2(1-\sigma)\left(\varepsilon_1\varepsilon_2 - \frac{1}{4}\varepsilon_3^2\right) + \frac{h^2}{3}[(\varkappa_1 + \varkappa_3)^2 - 2(1-\sigma)(\varkappa_1\varkappa_2 - \varkappa_3^2)]$$

$$\varepsilon_1 = \frac{1}{A}\frac{\partial u}{\partial\alpha} + \frac{1}{AB}\frac{\partial A}{\partial\beta}v - \frac{w}{R_1}, \quad \varepsilon_2 = \frac{1}{B}\frac{\partial w}{\partial\beta} + \frac{1}{AB}\frac{\partial B}{\partial\alpha}u - \frac{w}{R_2}$$

$$\varepsilon_3 = \frac{A}{B}\frac{\partial}{\partial\beta}\left(\frac{u}{A}\right) + \frac{B}{A}\frac{\partial}{\partial\alpha}\left(\frac{v}{B}\right)$$

$$\varkappa_1 = -\frac{1}{A}\left(\frac{\partial\gamma}{\partial\alpha} + \frac{\gamma'}{B}\frac{\partial A}{\partial\beta}\right), \quad \varkappa_2 = -\frac{1}{B}\left(\frac{\partial\gamma'}{\partial\beta} + \frac{\gamma}{A}\frac{\partial B}{\partial\alpha}\right)$$

$$\varkappa_3 = -\frac{1}{A}\frac{\partial\gamma'}{\partial\alpha} + \frac{\gamma}{AB}\frac{\partial A}{\partial\beta} + \frac{1}{R_1}\left(\frac{1}{B}\frac{\partial u}{\partial\beta} - \frac{1}{AB}\frac{\partial B}{\partial\alpha}v\right)$$

$$R_1^{-1} = -(\mathbf{n}\cdot\mathbf{M}_{\alpha\alpha})A^{-2}, \quad R_2^{-1} = -(\mathbf{n}\cdot\mathbf{M}_{\beta\beta})B^{-2}$$

Here E is the modulus of elasticity, σ ($\sigma < 1$) is the Poisson's ratio, $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and $\varkappa_1, \varkappa_2, \varkappa_3$ are the components of the tangential and bending deformation respectively, and R_1, R_2 are the principal radii of curvature of the surface S.

Let us denote by $\Pi_r(q_1, \ldots, q_{n-1})$ the potential energy of the forces acting on the rigid body, and by $U(\xi_1, \xi_2, \xi_3)$ the force functon of the mass forces applied to the particles of the shell and the fluid. We denote by $U(x_1, x_2, x_3, q_1, \ldots, q_{n-1})$ the force function transformed to the variables x_1, x_2, x_3. Then we have the following expression for the potential energy of the external forces:

$$\Pi_e = \Pi_r - \rho_1\int_S U_*(\alpha, \beta; u, v, w, \gamma, \gamma', q_1, \ldots, q_{n-1})\, AB\, d\alpha\, d\beta - \rho_2\int_\tau U\, d\tau \tag{1.7}$$

$$U_* = \int_{-h}^{h} U(x_1^* + w_1, x_2^* + w_2, x_3^* + w_3; q_1, \ldots, q_{n-1}) \times \left(1 + \frac{z}{R_1}\right)\left(1 + \frac{z}{R_2}\right)dz$$

where ρ_1 and ρ_2 denote the densities of the shell and fluid, and τ is the region occupied by the fluid at the particular instant.

The mechanical system under consideration admits the energy integral $T + \Pi = \text{const}$ ($\Pi = \Pi_e + \Pi_d$) and area integrals $\mathbf{G}_{O'}\cdot\xi_3^\circ = k = \text{const}$ where T and $\mathbf{G}_{O'}$ denote the kinetic energy and kinetic moment of the system relative to the point O' and ξ_3° is the unit vector of the ξ_3 axis, the projections of which on the x_i axes are denoted by $\nu_i = \nu_i(q_1, \ldots, q_{n-1})(i = 1, 2, 3)$. We introduce the rectangular $O'\xi_1'\xi_2'\xi_3$ coordinate system rotating about the ξ_3 axis with angular velocity Ω, and denote by T^* and $\mathbf{G}_{O'}^*$ the kinetic energy and kinetic moment of the system relative to the point O' in its motion relative to the $O'\xi_1'\xi_2'\xi_3$ axes. Then the energy area integrals become

$$T^* + \Omega\mathbf{G}_{O'}^*\cdot\xi_3^\circ + {}^1/_2 J\Omega^2 + \Pi = \text{const}, \quad \mathbf{G}_{O'}^*\cdot\xi_3^\circ + J\Omega = k$$

where J is the moment of inertia of the system about the ξ_3 axis. The quantity Ω is chosen so that the relation $\mathbf{G}_{O'}^*\xi_3^\circ = 0$ holds at any instant of time. Then $J\Omega = k$ and the energy integral can be written in the form $T^* + W = \text{const}$, where

$$W = \frac{k^2}{2J} + \Pi \tag{1.8}$$

is the changed potential energy of the system. Taking into account (1.4), we obtain the following expression for J:

$$J = \sum_{(123)} \{J_1 \nu_1^2 + M[(\nu_2 X_3 - \nu_3 X_2)^2 + 2R^{(1)} x_{1C}]\} + \tag{1.9}$$

$$4\rho_1 h \int_S J_*(\alpha, \beta; u, v, w, \gamma, \gamma'; q_1, \ldots, q_{n-1})\, AB\, d\alpha\, d\beta + \rho_2 \int_\tau (\xi_1^2 + \xi_2^2)\, d\tau$$

$$M x_{iC} = 2\rho_1 h \int_S \left[\left(1 + \frac{h^2}{3R_1R_2}\right) w_i^{(0)} - \frac{1}{3}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) h^2 w_i^{(1)}\right] AB\, d\alpha\, d\beta$$

$$2J_* = \left(1 + \frac{h^2}{3R_1R_2}\right) \sum_{(123)} [\nu_1^2 (2x_2 w_2^{(0)} + 2x_3 w_3^{(0)} + w_2^{(0)^2} + w_3^{(0)^2}) -$$

$$2\nu_2\nu_3 (x_2 w_3^{(0)} + x_3 w_2^{(0)} + w_2^{(0)} w_3^{(0)})] + \frac{1}{3} h^2 \sum_{(123)} [\nu_1^2 (w_2^{(1)^2} +$$

$$w_3^{(1)^2} - 2n_2 w_2^{(1)} - 2n_3 w_3^{(1)}) + 2\nu_2\nu_3 (x_2 w_3^{(1)} + n_3 w_2^{(1)} - w_2^{(1)} w_3^{(1)})] -$$

$$\frac{2}{3} h^2 \left(\frac{1}{R_1} + \frac{1}{R_2}\right) \sum_{(123)} [\nu_1^2 (x_2 w_2^{(1)} + x_3 w_3^{(1)} -$$

$$n_2 w_2^{(0)} - n_3 w_3^{(0)} + w_2^{(0)} w_2^{(1)} + w_3^{(0)} w_3^{(1)}) + \nu_2\nu_3 (n_2 w_3^{(0)} +$$

$$n_3 w_2^{(0)} - x_2 w_3^{(1)} - x_3 w_2^{(1)} - w_2^{(0)} w_3^{(1)} - w_3^{(0)} w_2^{(1)})]$$

Here M, J_1, J_2, J_3 are the mass and the principal axes of inertia of the rigid body and shell in its undeformed state, $X_i(q_1, \ldots, q_{n-1})$ $(i = 1, 2, 3)$ are the projections on the x_i axes of the radius vector drawn from the point O' to the point O, $R^{(i)}$ are the projections on the same axes of the vector $\mathbf{R} = \mathbf{X} - \boldsymbol{\nu}(\mathbf{X}\cdot\boldsymbol{\nu})$ describing the shortest distance between the ξ_3 axis and the point O, and x_{iC} are the coordinates of the center of mass of the rigid body and shell.

2. We obtain the equations of steady motion from the principle of virtual displacements, calculating the first variation δW and equating it to the elementary work δA_p done over the virtual displacements by the forces of external $p^{(+)}$ and internal $p^{(-)}$ gas pressure acting on the shell. The equations have the form

$$\frac{\partial W}{\partial q_j} \equiv \frac{\partial \Pi_e}{\partial q_j} - \frac{1}{2}\Omega^2 \frac{\partial J}{\partial q_j} = 0 \qquad (j = 1, \ldots, n-1) \tag{2.1}$$

$$\operatorname{grad}\left[U + \frac{1}{2}\Omega^2(\xi_1^2 + \xi_2^2) - \frac{p}{\rho_2}\right] = 0 \text{ in } \tau \tag{2.2}$$

$$p = p^{(-)} \text{ on } \Sigma \tag{2.3}$$

$$\frac{\partial \Pi_*}{\partial u} - D_\alpha(u) - D_\beta(u) - \frac{(1-\sigma^2)\rho_1}{2Eh}\left(\frac{\partial U_*}{\partial u} - \frac{1}{R_1}\frac{\partial U_*}{\partial \gamma}\right) - \tag{2.4}$$

$$\frac{(1-\sigma^2)\rho_1\Omega^2}{E}\left\{\frac{\partial J_*}{\partial u} - \frac{1}{R_1}\frac{\partial J_*}{\partial \gamma} + \right.$$

$$\left.\frac{1}{A}\left[1 + \frac{h^2}{3R_1}\left(\frac{1}{R_1} + \frac{2}{R_2}\right)\right] \sum_{(123)} x_{1\alpha} R^{(1)}\right\} = 0$$

$$\frac{\partial \Pi_*}{\partial w} - D_\alpha(w) - D_\beta(w) + D_{\alpha\alpha}(w) + D_{\alpha\beta}(w) + D_{\beta\beta}(w) -$$

$$\frac{1}{2} D(U_*) - h\Omega^2 \left\{D(J_*) - \sum_{(123)} \left\{\left(1 + \frac{h^2}{3R_1R_2}\right) n_1 + \right.\right.$$

$$\left.\left.\frac{h^2}{3AB}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\left[\frac{\partial}{\partial\alpha}\left(\frac{B}{A} x_{1\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{A}{B} x_{1\beta}\right)\right] R^{(1)}\right\}\right\} =$$

$$\begin{cases} F(p) & \text{on } S_1 \\ F(p^{(-)}) & \text{on } S_2 \end{cases}$$

$$D_\lambda(f) = \frac{1}{AB}\frac{\partial}{\partial\lambda}\left(AB\frac{\partial \Pi_*}{\partial f_\lambda}\right), \quad D_{\mu\nu}(f) = \frac{1}{AB}\frac{\partial^2}{\partial\mu\,\partial\nu}\left(AB\frac{\partial \Pi_*}{\partial f_{\mu\nu}}\right)$$

$$D(V) = \frac{(1-\sigma^2)\rho_1}{Eh}\left\{\frac{\partial V}{\partial w} + \frac{1}{AB}\left[\frac{\partial}{\partial\alpha}\left(B\frac{\partial U_*}{\partial\gamma}\right) + \frac{\partial}{\partial\beta}\left(A\frac{\partial U_*}{\partial\gamma'}\right)\right]\right\}$$

$$F(p) = \frac{1-\sigma^2}{2Eh}\left[p^{(+)}\left(1 + \frac{h}{R_1}\right)\left(1 + \frac{h}{R_2}\right) - p\left(1 - \frac{h}{R_1}\right)\left(1 - \frac{h}{R_2}\right)\right]$$

Here Σ is the free surface of the fluid, S_1 and S_2 are the parts of S corresponding to the parts of the cavity walls wetted and not wetted by the fluid, $S_1 + S_2 = S$, and p is the hydrodynamic pressure. The equations (2.1) — (2.4) must be supplemented by an equation obtained from the first equation of (2.4) by replacing $u, \alpha, A, B, R_1, R_2, \gamma$ by $v, \beta, B, A, R_2, R_1, \gamma'$ respectively, and adding the condition (1.5).

The steady motions represent uniform rotations of the system regarded as a rigid body about the ξ_3 axis, with the angular velocity $\Omega = kJ_0^{-1}$, where J_0 is the value of J for the steady motion. Integrating (2.2) and using (2.3), we find the pressure within the fluid and equation of its free surface during the steady motion

$$p(\xi_1, \xi_2, \xi_3) = \rho_2 U(\xi_1, \xi_2, \xi_3) + {}^1/_2\rho_2 \Omega^2 (\xi_1^2 + \xi_2^2) - \rho_2 c' \tag{2.5}$$

$$U(\xi_1, \xi_2, \xi_3) + {}^1/_2\Omega^2 (\xi_1^2 + \xi_2^2) = c' + p^{(-)}/\rho_2 \tag{2.6}$$

The value of the constant c' is determined by the amount of fluid within the cavity. Equations (2.4) with (2.5) and (1.5) taken into account are used to determine the deformation of the shell in the course of steady motion. Having found the form of the free fluid surface and shell deformation, we obtain from (2.1) the values $q_1, \ldots, q_{n-1}$ for the steady motion of the system.

3. Let us consider a certain steady motion. To simplify the calculations we assume that in this motion $q_j = 0$ $(j = 1, \ldots, n-1)$ and the whole deformed part of the cavity surface is wetted by the fluid. Let

$$q_j = 0 \ (j = 1, \ldots, n-1), \quad u = u_0(\alpha, \beta), \ v = v_0(\alpha, \beta), \ w = w_0(\alpha, \beta) \tag{3.1}$$

be a particular solution of the equations of steady motion for which the fluid occupies the region τ_0 bounded by the free surface Σ_0 determined by the equation

$$\Phi_0(\xi_1, \xi_2, \xi_3) = U(\xi_1, \xi_2, \xi_3) + {}^1/_2\Omega^2 (\xi_1^2 + \xi_2^2) = c_0 \ (c_0 = c_0' + p^{(-)}/\rho_2) \tag{3.2}$$

by the inner surface $S_0^{(-)}$ of the shell, and by the part Σ_0', wetted by the fluid, of the surface of the undeformed cavity walls. The position of the fluid in relation to the surface (3.2) is on that side, for which $\Phi_0 \geqslant c_0$.

Let us investigate the stability of the motion (3.1) (the definition of stability is given in /2/). We obtain the conditions of stability from the V.V. Rumiantsev theorem /3/ as the sufficient condition for the minimum of the changed potential energy W for the motion (3.1). Let us put, in the perturbed motion,

$$u = u_0 + u_*, \ v = v_0 + v_*, \ w = w_0 + w_* \tag{3.3}$$

and retain the previous notation for q_j. We denote the quantities $\gamma, \gamma', \varepsilon_i, \varkappa_i, x_{iC}$ corresponding to the values of (3.3) by

$$\gamma = \gamma_0 + \gamma_*, \ \gamma' = \gamma_0' + \gamma_*', \ \varepsilon_i = \varepsilon_{i0} + \varepsilon_{i*}$$
$$\varkappa_i = \varkappa_{i0} + \varkappa_{i*}, \ x_{iC} = x_{iC}^c + x_{iC}^* \tag{3.4}$$

where the quantities with zero subscript correspond to the motion (3.1).

From (1.8) we obtain the expression for the second variation

$$\delta^2 W = -{}^1/_2\, \Omega^2 \delta^2 J + \delta^2 \Pi + \Omega^2 J_0^{-1} (\delta J)^2 \tag{3.5}$$

In addition to the surface (3.2), we introduce a two-parameter family of surfaces

$$\Phi_1 = U(\xi_1, \xi_2, \xi_3) + \frac{k^2}{2(J_0 + \delta J)^2} (\xi_1^2 + \xi_2^2) = c_0 + \Delta c \tag{3.6}$$

Let us consider, for some sufficiently small values of q_j $(j = 1, \ldots, n-1)$, u_*, v_*, w_*, the region τ_1 occupied by the fluid in the case when its free surface belongs to the family (3.6), with Δc determined from the condition that the values of regions volume τ_0 and τ_1 are the same. We compute the variation $\Delta W = {}^1/_2 (\delta^2 W)_{\tau=\tau_1}$ of the functional W occurrring when the system passes from the unperturbed state (3.1) to another, sufficiently close perturbed state in which the fluid occupies the region τ_1. The transformation is carried out in two stages /4/: 1) the whole system regarded as a rigid body is displaced into the perturbed state; 2) the shell is deformed by imparting to it additional small elastic displacements u_*, v_*, w_* and the fluid is deformed by applying to its boundary surface a layer $\Delta\tau = \tau_1 - \tau_0$ of zero volume, into the shape with free surface (3.6) so that the fluid occupies the region τ_1. We write the expression for $(\delta^2 W)_{\tau=\tau_1}$ in the form

$$(\delta^2 W)_{\tau=\tau_1} = \delta_1^2 W + \delta_2^2 W, \quad \delta_2^2 W = \delta_{2(1)}^2 W + \delta_{2(2)}^2 W \tag{3.7}$$

Here $\delta_1^2 W$ and $\delta_2^2 W$ denote the increments in W incurred during the passage of the system to its perturbed state as a rigid body, and when the shell is deformed and a layer $\Delta\tau$ placed on the boundary surface of the fluid respectively, $\delta_{2(1)}^2 W$ and $\delta_{2(2)}^2 W$ are the parts of the increment $\delta_2^2 W$ not depending and depending respectively on the presence of the fluid. Similarly we have

$$\delta J = \delta_1 J + \delta_2 J,\ \delta_2 J = \delta_{2(1)} J + \delta_{2(2)} J \qquad (3.8)$$

$$\delta^2 J = \delta_1{}^2 J + \delta_2{}^2 J,\ \delta_2{}^2 J = \delta^2_{2(1)} J + \delta^2_{2(2)} J$$

Taking into account (3.5), (3.7) and (3.8) we can write the increments $\delta^2_{2(1)} W$ and $\delta^2_{2(2)} W$ in the form

$$\delta^2_{2(1)} W = \delta^2_{2(1)} \Pi_d + \delta^2_{2(1)} \Pi_e - \frac{1}{2} \Omega^2 \delta^2_{2(1)} J + \Omega^2 J_0^{-1} [2\delta_1 J \delta_{2(1)} J + (\delta_{2(1)} J)^2] \qquad (3.9)$$

$$\delta_{2(2)} W = \delta_{2(2)} \Pi_e - \frac{1}{2} \Omega^2 \delta_{2(2)} J + \Omega^2 J_0^{-1} [2(\delta_1 J + \delta_{2(1)} J)\delta_{2(2)} J + (\delta_{2(2)} J)^2] \qquad (3.10)$$

From (1.8) and (1.9) we find

$$\delta_1{}^2 W = \sum_{i,j=1}^{n-1} \left(\frac{\partial^2 W}{\partial q_i \partial q_j}\right)_0 q_i q_j, \quad \delta_1 J = \sum_{j=1}^{n-1} \left(\frac{\partial J}{\partial q_j}\right)_0 q_j \qquad (3.11)$$

Integrating (1.9) by parts and taking into account (1.3) and (1.5), we obtain

$$\delta_{2(1)} J = 4\rho_1 h \int_S \Big\{ \sum\nolimits^* \Big\{ \frac{\partial J_*}{\partial u} - \frac{1}{R_1}\frac{\partial J_*}{\partial \gamma} + \frac{1}{A}\Big[1 + \frac{h^2}{3R_1}\Big(\frac{1}{R_1} + \frac{2}{R_2}\Big)\Big] \times \qquad (3.12)$$
$$\sum_{(123)} x_{1\alpha} R^{(1)}\Big\}_0 u_* + \Big\{\frac{\partial J_*}{\partial w} + \frac{1}{AB}\Big[\frac{\partial}{\partial \alpha}\Big(B\frac{\partial J_*}{\partial \gamma}\Big) + \frac{\partial}{\partial \beta}\Big(A\frac{\partial J_*}{\partial \gamma'}\Big)\Big] -$$
$$\sum_{(123)} \Big[\Big(1 + \frac{h^2}{3R_1R_2}\Big) n_1 + \frac{h^2}{3AB}\Big(\frac{1}{R_1} + \frac{1}{R_2}\Big) \times$$
$$\Big(\frac{\partial}{\partial \alpha}\frac{B}{A} x_{1\alpha} + \frac{\partial}{\partial \beta}\frac{A}{B} x_{1\beta}\Big)\Big] R^{(1)}\Big\}_0 w_*\Big\} AB\, d\alpha\, d\beta$$

Here and henceforth Σ^* denotes a sum of two similar expressions, the second of which is obtained from the first when $u, \gamma, \alpha, A, R_1, R_2$ are replaced by $v, \gamma', \beta, B, R_2, R_1$ respectively. Further, (1.7), (1.8), (1.4) and (3.3) yield, after integrating by parts ($\gamma_1 = \gamma,\ \gamma_2 = \gamma'$)

$$\delta^2_{2(1)} \Pi_e = -\rho_1 \int_S \Big\{ 2\sum_{j=1}^{n-1} \Big[\sum\nolimits^* \Big(\frac{\partial^2 U_*}{\partial u\, \partial q_j} - \frac{1}{R_1}\frac{\partial^2 U_*}{\partial \gamma\, \partial q_j}\Big)_0 u_* + \qquad (3.13)$$
$$\Big(\frac{\partial^2 U_*}{\partial w \partial q_j} + \frac{1}{AB}\frac{\partial}{\partial \alpha}\Big(B\frac{\partial^2 U_*}{\partial \gamma \partial q_j}\Big) + \frac{1}{AB}\frac{\partial}{\partial \beta}\Big(A\frac{\partial^2 U_*}{\partial \gamma' \partial q_j}\Big)\Big)_0 w_*\Big] q_j +$$
$$\sum_{(uvw)} \Big(\frac{\partial^2 U_*}{\partial u \partial v}\Big)_0 u_* v_* + 2\sum_{i=1}^{2}\sum_{(uvw)} \Big(\frac{\partial^2 U_*}{\partial u \partial \gamma_i}\Big)_0 \gamma_{i*} u_* +$$
$$\sum_{i,j=1}^{2} \Big(\frac{\partial^2 U_*}{\partial \gamma_i \partial \gamma_j}\Big)_0 \gamma_{i*}\gamma_{j*}\Big\} AB\, d\alpha\, d\beta$$

Similarly, from (1.9) we obtain

$$\delta^2_{2(1)} J = 4M \sum_{j=1}^{n-1}\sum_{(123)} \Big(\frac{\partial R^{(1)}}{\partial q_j}\Big)_0 q_j x^*_{1C} + 4\rho_1 h \int_S \Big\{ 2\sum_{j=1}^{n-1} \Big[\sum\nolimits^* \times \qquad (3.14)$$
$$\Big(\frac{\partial^2 J_*}{\partial u \partial q_j} - \frac{\partial^2 J_*}{\partial \gamma \partial q_j}\Big)_0 u_* + \Big(\frac{\partial^2 J_*}{\partial w \partial q_j} + \frac{1}{AB}\frac{\partial}{\partial \alpha} B\frac{\partial^2 J_*}{\partial \gamma \partial q_j} +$$
$$\frac{1}{AB}\frac{\partial}{\partial \beta}\frac{\partial^2 J_*}{\partial \gamma' \partial q_j}\Big)_0 w_*\Big] q_j + \sum_{(uvw)} \Big(\frac{\partial^2 J_*}{\partial u \partial v}\Big)_0 u_* v_* +$$
$$2\sum_{i=1}^{2}\sum_{(uvw)} \Big(\frac{\partial^2 J_*}{\partial u \partial \gamma_i}\Big)_0 u_* \gamma_{i*} + \sum_{i,j=1}^{2} \Big(\frac{\partial^2 J_*}{\partial \gamma_i \partial \gamma_j}\Big)_0 \gamma_{i*}\gamma_{j*}\Big\} AB\, d\alpha\, d\beta$$

$$M x^*_{iC} = 2\rho_1 h \int_S \Big\{ \sum\nolimits^* \Big[1 + \frac{h^2}{3R_1}\Big(\frac{1}{R_1} + \frac{2}{R_2}\Big)\Big] \frac{x_{i\alpha}}{A} u_* - \qquad (3.15)$$
$$\Big[\Big(1 + \frac{h^2}{3R_1R_2}\Big) n_1 + \frac{h^2}{3AB}\Big(\frac{\partial}{\partial \alpha}\frac{B}{A}\Big(\frac{1}{R_1} + \frac{1}{R_2}\Big) x_{i\alpha} +$$
$$\frac{\partial}{\partial \beta}\frac{A}{B}\Big(\frac{1}{R_1} + \frac{1}{R_2}\Big) x_{i\beta}\Big)\Big] w_*\Big\} AB\, d\alpha\, d\beta$$

while from (1.8) and (3.4) we have

$$\delta^2_{2(1)} \Pi_d = \frac{4Eh}{1 - \sigma^2} \int_S \Pi_* (\varepsilon_*, \varkappa_*)\, AB\, d\alpha\, d\beta \qquad (3.16)$$

In this manner (3.11)— (3.16) yield the expression (3.9) for $\delta^2_{2(1)}W$.

Let us now find $\delta^2_{2(2)}W$. Taking into account (1.7) and (1.9) we transform the expression (3.10) to the form

$$\delta^2_{2(2)} W = -2\rho_2 \int_{\Delta\tau} \left[U(\xi_1, \xi_2, \xi_3) + \frac{1}{2}\Omega^2(\xi_1^2 + \xi_2^2) \right] d\tau + \Omega^2 J_0^{-1} [2(\delta_1 J + \delta_{2(1)} J)\delta_{2(2)} J + (\delta_{2(2)} J)^2] \tag{3.17}$$

Here the integral of the function over $\Delta\tau$ should be regarded as a difference of the integrals of this function over the regions τ_1 and τ_0. Let us denote by $\Phi(x_1, x_2, x_3, q_j)$ the integrand function in (3.17) transformed to the variables x_1, x_2, x_3. Equations (3.2) and (3.6) in these variables will assume the form

$$\Phi_0 \equiv \Phi(x_1, x_2, x_3, 0) = c_0, \quad \Phi_1(x_1, x_2, x_3, q_j, \delta J) = c_0 + \Delta c \tag{3.18}$$

The equation for Φ_1 is written with the accuracy of up to the terms of first order of smallness, in the form

$$\Phi_1 = \Phi(x_1, x_2, x_3, q_j) - \Omega^2 J_0^{-1}(\xi_1^2 + \xi_2^2)\delta J = c_0 + \Delta c \tag{3.19}$$

The fluid in the regions τ_0 and τ_1 is situated, with respect to the surfaces (3.18), on the side for which $\Phi_0 \geqslant c_0, \Phi_1 \geqslant c_0 + \Delta c$ respectively.

Let N_0 be a certain point on the surface $\Phi_0 = c_0$ while N be a point belonging to the space (x_1, x_2, x_3) and lying on the perpendicular to this surface passing through the point N_0 and sufficiently close to it. We denote by $\mathbf{x}_0$ and $\mathbf{x}$ the radius vectors of the points N_0 and N relative to O. Then

$$\Delta\mathbf{x} = \mathbf{x} - \mathbf{x}_0 = -\lambda \operatorname{grad} \Phi_0 \tag{3.20}$$

where λ is the proportionality factor, positive for the point $N: \Phi_0 < c_0$ and negative if $\Phi_0 > c_0$. We denote by λ_1 the value of λ for the surface (3.19). We have, with the accuracy of up to the first order of smallness

$$\lambda_1 |\operatorname{grad} \Phi_0|_*^2 = \sum_{j=1}^{n-1} \left(\frac{\partial\Phi}{\partial q_j}\right)_* q_j - \Omega^2 J_0^{-1}(\xi_1^2 + \xi_2^2)_* \delta J - \Delta c \tag{3.21}$$

where the asterisk means that the corresponding quantity is computed on the surface $\Phi_0 = c_0$ for $q_j = 0$ $(j = 1, \ldots, n-1)$.

Let us now consider the points of the space (x_1, x_2, x_3) lying near the inner surface $S_0^{(-)}$ of the shell, for the unperturbed state of the system. We denote by $\mathbf{r}_0, \mathbf{r}_1$ and $\mathbf{r}$ the radius vectors, relative to the point O of the inner surface of the shell for the unperturbed and perturbed state of the system, and of the point lying near the surface $S_0^{(-)}$. From (1.3) we obtain the following expressions for these radius vectors:

$$\mathbf{r}_0 = \mathbf{M} + \mathbf{U}_0 - h(\mathbf{n} + \mathbf{n} \times \mathbf{\Omega}_0), \quad \mathbf{r}_1 = \mathbf{r}_0 + \mathbf{U}_* - h(\mathbf{n} \times \mathbf{\Omega}_*), \quad \mathbf{r} = \mathbf{r}_0 + z(\mathbf{n} + \mathbf{n} \times \mathbf{\Omega}_0) \tag{3.22}$$

From these we find, taking into account (1.4), the following expressions for the coordinates x_1, x_2 and x_3 of the vector $\mathbf{r}$:

$$x_i = x_i(\alpha, \beta) + w_{i0}^{(0)} - (h - z)(n_i - w_{i0}^{(1)}) \quad (i = 1, 2, 3) \tag{3.23}$$

where $w_{i0}^{(0)}$ and $w_{i0}^{(1)}$ are the values of $w_i^{(0)}$ and $w_i^{(1)}$ for the unperturbed motion. We denote by z_1 the value of z for the inner surface $S^{(-)}$ of the shell in the system in perturbed state. From (3.22) and (1.3) we obtain, with the accuracy of up to the terms of first order of smallness,

$$z_1 = (\mathbf{r}_1 - \mathbf{r}_0) \cdot (\mathbf{n} + \mathbf{n} \times \mathbf{\Omega}_0) = \mathbf{U}_* \cdot \mathbf{n} = -w_* \tag{3.24}$$

The condition that the values of regions volume τ_0 and τ_1 are equal and (3.21), (3.24), together yield the following linear equation with the accuracy of up to terms of first order of smallness:

$$\int_{\Sigma_0} \left\{ \Delta c + \Omega^2 J_0^{-1}(\xi_1^2 + \xi_2^2)_* \delta J - \sum_{1=j}^{n-1} \left(\frac{\partial\Phi}{\partial q_j}\right)_* q_j \right\} \frac{dS}{|\operatorname{grad}\Phi_0|} - \int_S w_* \left(1 - \frac{h}{R_1}\right)\left(1 - \frac{h}{R_2}\right) AB \, d\alpha \, d\beta = 0 \tag{3.25}$$

which connects Δc with $\delta_{2(2)}J$. Another similar equation can be obtained by calculating, in the first approximation, the quantity

$$\delta_{2(2)} J = \rho_2 \int_{\Delta\tau} (\xi_1^2 + \xi_2^2) \, d\tau = \rho_2 \int_{\Sigma_0} \left\{ \Delta c + \Omega^2 J_0^{-1}(\xi_1^2 + \xi_2^2) \delta J - \right. \tag{3.26}$$

$$\sum_{j=1}^{n-1}\left(\frac{\partial\Phi}{\partial q_j}\right)_* q_j\Big\}\frac{|\xi_1^2+\xi_2^2|_*}{|\text{grad}\,\Phi_0|}\,dS-\rho_2\int_S w_*(\xi_1^2+\xi_2^2)_0\times$$
$$\left(1-\frac{h}{R_1}\right)\left(1-\frac{h}{R_2}\right)AB\,d\alpha\,d\beta$$

Taking into account (3.8), (3.11) and (3.12) we obtain, from (3.25) and (3.26), the following expressions for Δc and $\delta_{2(2)}J$:

$$\Delta c=Q^{(1)},\ \delta_{2(2)}J=Q^{(2)} \tag{3.27}$$
$$Q^{(i)}=\sum_{j=1}^{n-1}e_j^{(i)}q_j+\int_S[f^{(i)}(\alpha,\beta)\,u_*+g^{(i)}(\alpha,\beta)v_*+h^{(i)}(\alpha,\beta)\,w_*]\,AB\,d\alpha\,d\beta$$
$$(i=1,2)$$

where $e_j^{(i)}$ denote the specified constants and $f^{(i)}, g^{(i)}, h^{(i)}$ are known functions of α and β. Let us now find the integral in (3.17)

$$\int_{\Delta\tau}\Phi(\xi_1,\xi_2,\xi_3)\,d\tau=\frac{1}{2}\int_{\Sigma_0}\Big\{[\Delta c+\Omega^2J_0^{-1}(\xi_1^2+\xi_2^2)_*\,\delta J]^2- \tag{3.28}$$
$$\Big[\sum_{j=1}^{n-1}\left(\frac{\partial\Phi}{\partial q_j}\right)_* q_j\Big]^2\frac{dS}{|\text{grad}\,\Phi_0|}+\frac{1}{2}\int_S\Big\{w_*^2\sum_{i=1}^{3}\left(\frac{\partial\Phi}{\partial x_i}\right)_0(n_i-w_{i0}^{(1)})-$$
$$2w_*\sum_{j=1}^{n-1}\left(\frac{\partial\Phi}{\partial q_j}\right)_0 q_j\Big\}\left(1-\frac{h}{R_1}\right)\left(1-\frac{h}{R_2}\right)AB\,d\alpha\,d\beta$$

From (3.17), (3.28), (3.11), (3.12) and (3.27) follows the expression for $\delta^2_{2(2)}W$ and the results obtained yield the following expressions for $\delta^2_{2(1)}W$ and $\delta W^2_{2(2)}$:

$$\delta^2_{2(1)}W=\delta^2_{2(1)}\Pi_d+\int_S[2\sum_{j=1}^{n-1}q_jL_{1j}(\alpha,\beta;u_*,v_*,w_*)+$$
$$V_2(\alpha,\beta;u_*,v_*,\gamma_*,\gamma_*')]\,AB\,d\alpha\,d\beta+\Omega^2J_0^{-1}I_1^2$$
$$\delta^2_{2(2)}W=\sum_{j=1}^{n-1}a_{ij}q_iq_j+\Big[2\sum_{j=1}^{n-1}q_jL_{2j}(\alpha,\beta;u_*,v_*,w_*)+$$
$$D(\alpha,\beta)\,w_*^2\Big]AB\,d\alpha\,d\beta-a_1I_2^2-2a_2I_2I_3-a_3I_3^2+\Omega^2J_0^{-1}(2I_1I_4+I_4^2)$$
$$I_k=\int_S L_k(\alpha,\beta;u_*,v_*,w_*)\,AB\,d\alpha\,d\beta,\quad k=1,2,3,4$$

Here $L_1,\ldots,L_4,L_{1j},L_{2j}$ are known linear forms of u_*,v_*,w_*, V_2 is a known quadratic form of u_*,v_*,w_*, γ_*,γ_*',D. is a known function of $\alpha,\beta;a_{ij}$ and a_1,a_2,a_3 are specified constants, with $a_i>0\ (i=1,2,3)$. Taking into account (3.7) and (3.11), we now finally obtain the following expression for $(\delta^2W)_{\tau=\tau_1}$:

$$(\delta^2W)_{\tau=\tau_1}=\sum_{j=1}^{n=1}g_{ij}q_iq_j+2\sum_{j=1}^{n-1}q_jI_j^{(1)}+\delta^2_{2(1)}\Pi_d+ \tag{3.29}$$
$$\int_S V^{(2)}AB\,d\alpha\,d\beta+\Omega^2J_0^{-1}I^2_{(1)}-a_1I_2^2-2a_2I_2I_3-a_3I_3^2$$
$$g_{ij}=a_{ij}+\left(\frac{\partial^2W}{\partial q_i\partial q_j}\right)_0,\qquad L_j^{(1)}=L_{1j}+L_{2j},\quad L_{(1)}=L_1+L_4,$$
$$V^{(2)}=V_2+Dw_*^2$$
$$I_j^{(1)}=\int_S L_j^{(1)}AB\,d\alpha\,d\beta,\qquad I_{(1)}=\int_S L_{(1)}AB\,d\alpha\,d\beta$$

When the system passes from the unperturbed state to the perturbed state sufficiently close to the unperturbed and the liquid occupies an arbitrary space τ for the given cavity, we have the following relation for δ^2W at fixed $q_j\ (j=1,\ldots,n-1)$ /2/:

$$\min_\tau\delta^2W=(\delta^2W)_{\tau=\tau_1} \tag{3.30}$$

4. Let the quadratic form of $q_1, \ldots, q_{n-1}$ in (3.29) be positive definite; in the case of the alternating signs of this form the unperturbed motion will be unstable in the temporal sense for the nondeformable shell when $u = v = w \equiv 0$. To simplify the calculations we shall assume that $g_{ij} = 0\ (i \neq j)$ and put $g_{ii} = g_i > 0$. We shall write the expression (3.29) in the form

$$(\delta^2 W)_{\tau=\tau_1} = (\delta^2 W)_* + \delta^2_{2(1)}\Pi_d + \int_S V^{(2)} AB\, d\alpha\, d\beta - \sum_{j=1}^{n-1} g_j^{-1} I_j^{(1)^2} - a_1 I_2^2 - 2a_2 I_2 I_3 - a_3 I_3^2 \tag{4.1}$$

$$(\delta^2 W)_* = \Omega^2 J_0^{-1} I_{(1)}^2 + \sum_{j=1}^{n-1} g_j (q_j + g_j^{-1} I_j^{(1)})^2 \geqslant 0$$

Let us consider the auxilliary variational problem of finding the minimum μ of the functional

$$F(u, v, w) = \frac{1-\sigma^2}{Eh} \Pi_d \left\{ \int_S [u^2 + v^2 + w^2 + h^2(\gamma^2 + \gamma'^2)]\, AB\, d\alpha\, d\beta \right\}^{-1} \tag{4.2}$$

where Π_d and γ, γ' are given by the formulas (1.8), (1.9) and (1.5), in the class of functions $u(\alpha, \beta)$, $v(\alpha, \beta)$, $w(\alpha, \beta)$, $0 \leqslant \alpha < 2\pi$, $\beta_1 \leqslant \beta \leqslant \beta_2$ with continuous up to the fourth order derivatives in α and β, satisfying the boundary conditions (1.7) and conditions of 2π-periodicity in α. Solving this problem we arrive at the inequality

$$\delta^2_{2(1)}\Pi_d \geqslant \frac{2\mu Eh}{1-\sigma^2} \int_S [u_*^2 + v_*^2 + w_*^2 + h^2(\gamma_*^2 + \gamma_*'^2)]\, AB\, d\alpha\, d\beta \tag{4.3}$$

Using the Cauchy—Buniakowski inequality we obtain from (3.30), (4.1) and (4.3) the following estimate:

$$\delta^2 W \geqslant (\delta^2 W)_* + \int_S V_*^{(2)}(\alpha, \beta; u_* v_*, \ , w_*, \gamma_*, \gamma_*')\, AB\, d\alpha\, d\beta \tag{4.4}$$

$$V_*^{(2)} = V^{(2)} + \frac{2\mu Eh}{1-\sigma^2}[u_*^2 + v_*^2 + w_*^2 + h^2(\gamma_*^2 + \gamma_*'^2)] - S\left[(a_1 + a_2) L_2^2 + (a_2 + a_3) L_3^2 + \sum_{j=1}^{-1} g_j^{-1} L_j^{(1)^2}\right]$$

where S is the area of the middle surface of the shell. From (4.4) we conclude that the inequalities $g_j > 0\ (j = 1, \ldots, n-1)$ and conditions of positive definiteness of the quadratic form $V_*^{(2)}$ relative to $u_*, v_*, w_*, \gamma_*, \gamma_*'$ together form the sufficient conditions for the positive definiteness of the functional $\delta^2 W$ and, by virtue of the Rumiantsev theorem /3/ they will represent the sufficient conditions of stability of the steady motion (3.1).

The above method of studying the stability of steady motions of mechanical systems containing perfectly rigid, elastic and fluid elements, was applied to practical problems in /2, 5—7/.

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